0915 Lecture 04

Tuesday, September 06, 2016 1:22 AM

Groupoids and equivariant (quasi-) coherent sheaves

What about non-free actions?

Still have $G \times X \xrightarrow{(p,\sigma)} X \times X$, this is the structure of a groupiod

$$X_1 \rightrightarrows X_0 = X \qquad (X_1 = G \times X)$$

groupiod: category where all morphisms are invertible

Morita maps between groupoids: kind of functor

$f : X_\bullet \to Y_\bullet$ s.t. $X_0 \xrightarrow{f_0} Y_0$ fppf (strong form of essential surjectivity) and

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_0 \times X_0 \arrow[d] \\ Y_1 \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$

(fully faithfullness)

NB groupoids form a 2-category

natural transformations are maps $\eta : X_0 \to Y_1$ satisfying...

Observation: Morita morph. w/ section induces

Observation: Morita morph. w/ section induces equivalence of categories

$f: X_\bullet \to Y_\bullet$ section on $Y_0 \to X_0$

induces $s: Y_\bullet \to X_\bullet$

Standard example: $G \circlearrowright X$ and $G \subset H$, then let $Y = H \times_G X = H \times X / G$

$$\begin{tikzcd} G \times X \arrow[d, shift left] \arrow[d, shift right] & H \times G \times X \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & H \times Y \arrow[d, shift left] \arrow[d, shift right] \\ X & H \times X \arrow[l] \arrow[r] & Y \end{tikzcd}$$

$\underbrace{\text{both morita morphisms}}_{\text{says } Y/H \cong X/G}$

Quasicoherent sheaves: on a groupoid $X_1 \rightrightarrows X_0$, consists of

1) quasicoherent sheaf on X_0, E
2) iso. $\alpha: t^*E \xrightarrow{\cong} s^*E$ such that

A) $X_1 \times_{X_0} X_1 \overset{p_2}{\underset{p_1}{\overset{c}{\Rrightarrow}}} X_1 \overset{s}{\underset{t}{\rightrightarrows}} X_0$

$(\gamma_2, \gamma_1) \longrightarrow (\gamma_1)$

$c^*t^*E \xrightarrow{\cong} c^*s^*E$

$\| \qquad\qquad \|$

$$(\gamma_0;\gamma_1) \to (\gamma_1),\quad (\gamma_0;\gamma_1) \to (\gamma_0\gamma_1),\quad (\gamma_0;\gamma_1) \to (\gamma_0)$$

$$p_1^* t^* E \qquad \text{commutes} \qquad p_2^* s^* E$$

B) $e^*\alpha : E \to E$ is the identity

__Ex:__ $QCoh(G \rightrightarrows \bullet) \cong Rep(G)$

__Example:__ $\mathbb{P}(V)$, the line bundle $\mathcal{O}(1)$.

Action of $PGL(V) \circlearrowright \mathbb{P}(V) \rightsquigarrow$ the line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ is not linearizable because under orbit map

$PGL(V) \longrightarrow \mathbb{P}(V)$ the pullback of $\mathcal{O}_{\mathbb{P}(V)}(1)$ has order $\dim(V)$ in $Pic(PGL(V))$

On other hand, $\mathcal{O}_{\mathbb{P}(V)}(1)$ is linearizable for action of $GL(V)$

↳ can see this geometrically,

or algebraically

or algebraically

Some key Facts:

1) the category $QCoh(X_\bullet)$ is abelian, with kernels & cokernels formed level wise
"if groupoid is flat"

2) the category has enough coherent sheaves (meaning...)

Application:

Recall Pic

Prop: if X is a projective k-variety s.t. $\bar{X}$ is normal, then it is a G-proj. / k-variety

Idea: show that if L is fixed as k-rational point of $Pic(X/k)$, then L^n is linearizable for some n

$$G \times X \underset{\sigma}{\overset{p_2}{\rightrightarrows}} X$$

idea: $Pic(G)$ is finite, so $p_2^* L \otimes \sigma^* L$, whose restr. to $\{1\} \times X$ is trivial, is trivial by a see-saw argument, after some power

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Can show that this iso. satisfies cocycle condition

G is rational, so must act trivially
(normality $\Rightarrow$ $\mathrm{Pic}(X/k)$ proper)

prove in exercises