

Groupoids and equivariant coherent sheaves (quasi-)

What about non-free actions?

Still have $G \times X \xrightarrow{(p, \sigma)} X \times X$, this is the structure of a groupoid
 $X_1 \rightrightarrows X_0 = X$
 category where all morphisms are invertible

Morita maps between groupoids: kind of functor

$$F: X_0 \rightarrow Y_0 \quad \text{s.t.} \quad X_0 \xrightarrow{f_0} Y_0 \quad \text{and} \quad X_1 \xrightarrow{f_1} X_0 \times X_0$$

(strong form of essential surjectivity)

$$Y_1 \xrightarrow{f_1} Y_0 \times Y_0 \quad \text{(fully faithfulness)}$$

NB groupoids form a 2-category
 natural transformations are maps
 $M: X_0 \rightarrow Y_0$ satisfying...

Observation: Morita morph. w/ section induces

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$f: X_0 \rightarrow Y_0$ section on $Y_0 \rightarrow X_0$ induces $S: Y_0 \rightarrow X_0$

Standard example: $G \curvearrowright X$ and $G \curvearrowright H$, then let $Y = H \times_G X = H \times X / G$

$$\begin{array}{ccccc}
 G \times X & \longleftarrow & H \times G \times X & \longrightarrow & H \times Y \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 X & \longleftarrow & H \times X & \longrightarrow & Y
 \end{array}$$

both morita morphisms
says $Y/H \cong X/G$

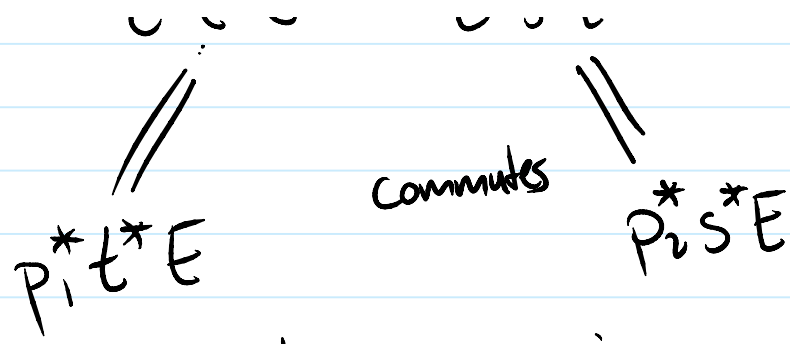
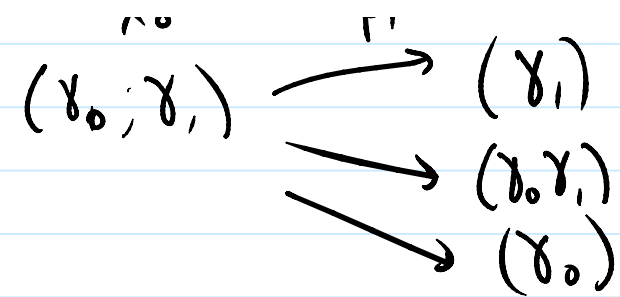
Quasicoherent sheaves: on a groupoid $X_1 \rightrightarrows X_0$, consists of

- 1) quasicoherent sheaf on X_0 , E
- 2) iso. $\alpha: t^*E \xrightarrow{\cong} s^*E$ such that

A)

$$\begin{array}{ccc}
 X_1 \times_{X_0} X_1 & \begin{array}{c} \xrightarrow{p_2} \\ \xrightarrow{c} \\ \xrightarrow{p_1} \end{array} & X_1 \xrightarrow[t]{s} X_0 \\
 (X_1, X_1) & \longrightarrow & (Y_1)
 \end{array}$$

$$\begin{array}{ccc}
 C^* t^* E & \xrightarrow{\cong} & C^* s^* E \\
 \parallel & & \parallel
 \end{array}$$



B) $e^* \alpha: E \rightarrow E$ is the identity

Ex: $\text{QEdn}(G \rightrightarrows \bullet) \cong \text{Rep}(G)$

Example: $\mathbb{P}(V)$, the line bundle $\mathcal{O}(1)$.

Action of $\mathbb{P}GL(V) \curvearrowright \mathbb{P}(V) \rightsquigarrow$ the line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ is not linearizable because under orbit map

$\mathbb{P}GL(V) \rightarrow \mathbb{P}(V)$ the pullback of $\mathcal{O}_{\mathbb{P}(V)}(1)$ has order $\dim(V)$ in $\text{Pic}(\mathbb{P}GL(V))$

On other hand, $\mathcal{O}_{\mathbb{P}(V)}(1)$ is linearizable for action of $GL(V)$

↳ can see this geometrically,
or algebraically.

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Some key facts:

if groupoid is flat

1) the category $\text{QCoh}(X_0)$ is abelian, with kernels & cokernels formed level wise

2) the category has enough coherent sheaves (meaning...)

Application:

Recall Pic

Prop: if X is a ^{projective} k -variety s.t. \bar{X} is normal, then it is a G -proj. k -variety

Idea: show that if L is fixed as k -rational point of $\text{Pic}(X/k)$, then L^n is linearizable for some n

$$G \times X \begin{array}{c} \xrightarrow{p_2} \\ \xrightarrow{\sigma} \end{array} X$$

idea: $\text{Pic}(G)$ is finite, so $p_2^* L \otimes \sigma^* L$, whose restr. to $\{1\} \times X$ is trivial, is trivial by a see-saw argument, after some power

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Can show that this iso. satisfies cocycle condition

G is rational, so must act trivially
(normality \Rightarrow $\text{Pic}(X/k)$ proper)
prove in exercises